

SOME ASYMPTOTIC PROPERTIES OF THE REES POWERS OF A MODULE

ANA L. BRANCO CORREIA AND SANTIAGO ZARZUELA

ABSTRACT. Let R be a commutative ring and let G be a free R -module with finite rank $e > 0$. For any R -submodule $E \subset G$ one may consider the image of the symmetric algebra of E by the natural map to the symmetric algebra of G , and then the graded components E_n , $n \geq 0$, of the image, that we shall call the n -th Rees powers of E (with respect to the embedding $E \subset G$). In this work we prove some asymptotic properties of the R -modules E_n , $n \geq 0$, which extend well known similar ones for the case of ideals, among them Burch's inequality for the analytic spread.

1. INTRODUCTION

Let R be a commutative ring and let $G \simeq R^e$ be a free R -module of finite rank $e > 0$. For any R -submodule E of G we may consider the natural graded morphism from the symmetric algebra of E to the symmetric algebra of G induced by the embedding of E in G , that we shall denote by $\phi : \mathcal{S}_R(E) \rightarrow \mathcal{S}_R(G)$. As in the case of ideals, it is natural in this situation to define the Rees algebra of E (with respect to the given embedding $E \subset G$) as the image of the morphism ϕ , namely $\mathcal{R}_G(E) := \phi(\mathcal{S}_R(E)) \subset \mathcal{S}_R(G)$. Note that the symmetric algebra of G is isomorphic to the polynomial ring $R[t_1, \dots, t_e]$ and that, by definition, $\mathcal{R}_G(G) = \mathcal{S}_R(G)$. We may now consider the graded components $[\mathcal{R}_G(E)]_n$, for $n \geq 0$, of $\mathcal{R}_G(E)$ that we shall call the n -th Rees powers of E (with respect to the given embedding $E \subset G$) and denote by E_n , for $n \geq 0$. In this way we may simply write $\mathcal{R}_G(E) = \bigoplus_{n \geq 0} E_n$. Observe that, by definition, we have that $E_n \subset G_n \simeq R^g$, with $g = \binom{n+e-1}{e-1}$ for $n \geq 0$.

Some general properties of the modules E_n , extending similar ones for ideals, have been proved for instance by D. Katz and C. Naude [12] and in the two dimensional case by D. Katz and V. Kodiyalam [11], V. Kodiyalam [13] and R. Mohan [16]. In particular, it is shown in [12] that if R is a Noetherian ring then the sets of associated primes $\text{Ass}(G_n/E_n)$ stabilize for $n \gg 0$, that is, $\text{Ass}(G_n/E_n) = \text{Ass}(G_{n+1}/E_{n+1})$ for $n \gg 0$, the corresponding result for ideals having been proved by M. Brodmann [1].

One of the most basic asymptotic properties of the powers of an ideal is given by the well known Burch's inequality relating the analytic spread of an ideal $I \subset R$ with the depths of the R -modules R/I^n , for $n \geq 0$. Recall that if (R, \mathfrak{m}) is a local ring of

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dimension d , then the analytic spread $\ell(I)$ of I may be defined as the dimension of the ring $\mathcal{F}(I) = \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$, the fiber cone (or special fiber) of I . It is well known that $\text{ht}(I) \leq \ell(I) \leq d$, and L. Burch [4] proved that $\ell(I) \leq d - \inf_{n \geq 1} \text{depth } R/I^n$. It is also known that this inequality becomes equality when the associated graded ring of I is Cohen-Macaulay, see for instance D. Eisenbud and C. Huneke [8].

Our main result in this paper is a natural extension to the case of modules of the Burch's inequality. Namely, we shall be able to prove the following result.

Theorem 1.1. *Let (R, \mathfrak{m}) be a local ring and let $G \simeq R^e$ be a free R -module of finite rank $e > 0$. Assume that $\dim R = d > 0$. Let $E \subset G$ be an R -submodule of G such that $E \neq G$ and denote by $\ell_G(E)$ the dimension of the graded ring $\mathcal{R}_G(E)/\mathfrak{m}\mathcal{R}_G(E)$. Then*

$$\ell_G(E) \leq d + e - 1 - \inf_{n \geq 1} \text{depth } G_n/E_n.$$

We may also prove that the above inequality becomes equality if the ring R and the Rees algebra $\mathcal{R}_G(E)$ are both Cohen-Macaulay. In the case of modules, there is a lack of a similar object as the associated graded ring of an ideal, but it is well known that if R is Cohen-Macaulay and $I \subset R$ is an ideal whose Rees algebra $\mathcal{R}(I)$ is Cohen-Macaulay, then the associated graded ring of I is Cohen-Macaulay too.

In order to prove Burch's inequality for modules we follow the approach by M. Brodmann [2] in the ideal case, which is based on the stable behavior of the depths of R/I^n , for $n > 0$. Roughly speaking, by using the result proven by D. Katz and C. Naude on the stability of the sets of associated primes of the R -modules G_n/E_n , for $n \gg 0$, we can first extend to the case of modules the result by M. Brodmann on the stable behavior of the depths of G_n/E_n , for $n \gg 0$, and then to prove Burch's inequality for modules.

Although in the case of ideals one can quickly prove Burch's inequality by using the associated graded ring, we are forced to follow this more complicated approach due to the lack of such a similar object in the theory of Rees algebras of modules.

As a final application we also extend to modules a famous criteria proved by R. C. Cowsik and M. V. Nori [7] for an ideal to be a complete intersection. A special case of complete intersection modules was already considered by D. A. Buchsbaum and D. Rim in [5] under the name of parameter matrices, playing the same role as system of parameters in their theory of multiplicities for modules of finite length (nowadays called Buchsbaum-Rim multiplicities). Other authors like D. Katz and C. Naude [12] have also studied the properties of complete intersection modules (or modules of the principal class).

Throughout this paper we shall always assume that (R, \mathfrak{m}) is a local ring of dimension d with maximal ideal \mathfrak{m} and that $G \simeq R^e$ is a free R -module with finite rank $e > 0$. For a given R -submodule $E \subset G$ of G we shall understand that an embedding of E in G has been fixed. Consequently, the Rees algebra $\mathcal{R}_G(E)$ of E , and so the n -th Rees powers of E , will always be with respect to this fixed embedding. One should note that, for a given R -module E and different embeddings of E into free R -modules, we can get non isomorphic Rees algebras of E , see for instance A.

Micali [15, Chapitre III, 2. Un exemple] or the more recent D. Eisenbud, C. Huneke and B. Ulrich [9, Example 1.1]. See also these papers for a discussion about the uniqueness of the definition of the Rees algebra of a module. In particular, it is known that if an R -module E has rank, that is, $E \otimes_R Q$ is free with positive rank, where Q is the total ring of fractions of R , then for any embedding $E \subset G \simeq R^e$ of E into a free R -module of positive rank, the Rees algebra of E (with respect to this embedding) is isomorphic to $\mathcal{S}_R(E)/T_R(\mathcal{S}_R(E))$, the symmetric algebra of E modulo its R -torsion, see [9].

We close this introduction with the following observation. Given $E \subset G \simeq R^e$ an R -module with $E \neq G$, the ideal of $\mathcal{R}_G(G)$ generated by E is the graded ideal

$$E\mathcal{R}_G(G) = E \oplus E \cdot G \oplus E \cdot G^2 \oplus \cdots = \bigoplus_{i \geq 0} E \cdot G_i.$$

For each $n \geq 1$,

$$(1) \quad (E\mathcal{R}_G(G))^n = \bigoplus_{i \geq 0} E_n \cdot G_i = E_n \mathcal{R}_G(G)$$

and so, in particular, $[E_n \mathcal{R}_G(G)]_n = E_n$.

2. THE ANALYTIC SPREAD OF A MODULE

Let $E \subset G$ be an R -submodule of G . If E has rank $e > 0$ there is an explicit formula for the dimension of the Rees algebra of E , namely $\dim \mathcal{R}_G(E) = d + e$, see for instance A. Simis, B. Ulrich and W. V. Vasconcelos [17, Proposition 2.2]. In the general case, there is not such an explicit formula, but one can prove that $\dim \mathcal{R}_G(E) \leq d + e$. For this we follow similar steps as in the proof of [17, Proposition 2.2]. First, we determine the set of minimal primes of the Rees algebra of E .

Lemma 2.1. *Let $E \subset G \simeq R^e$, $e > 0$, be an R -module. Then*

$$\text{Min } \mathcal{R}_G(E) = \{\mathcal{P} = \mathfrak{p}\mathcal{R}_G(G) \cap \mathcal{R}_G(E) \mid \mathfrak{p} \in \text{Min } R\}.$$

Proof. For any R -ideal J , we set $\mathcal{J} = J\mathcal{R}_G(G) \cap \mathcal{R}_G(E)$. It is easy to prove that if $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ is a shortest primary decomposition in R then $(0) = \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_s$ is a shortest primary decomposition in $\mathcal{R}_G(E)$. Hence

$$\text{Ass } \mathcal{R}_G(E) = \{\sqrt{\mathcal{Q}_1}, \dots, \sqrt{\mathcal{Q}_s}\} \supseteq \text{Min } \mathcal{R}_G(E).$$

Now, if $\mathcal{P} \in \text{Min } \mathcal{R}_G(E)$, $\mathcal{P} = \sqrt{\mathcal{Q}_i}$ for some $1 \leq i \leq s$. But

$$\sqrt{\mathcal{Q}_i} = \sqrt{\mathfrak{q}_i \mathcal{R}_G(G) \cap \mathcal{R}_G(E)} = \sqrt{\mathfrak{q}_i} \mathcal{R}_G(G) \cap \mathcal{R}_G(E)$$

(since $\mathcal{R}_G(G)$ is a polynomial ring) and, by minimality of $\sqrt{\mathcal{Q}_i}$, $\sqrt{\mathfrak{q}_i} \in \text{Min}(\text{Ass } R) = \text{Min } R$. For the other inclusion, let $\mathcal{P} = \mathfrak{p}\mathcal{R}_G(G) \cap \mathcal{R}_G(E)$ with $\mathfrak{p} \in \text{Min } R$. Then $\mathfrak{p} = \sqrt{\mathfrak{q}_i}$ for some $1 \leq i \leq s$ and we have $\mathcal{P} = \sqrt{\mathcal{Q}_i} = \sqrt{\mathfrak{q}_i \mathcal{R}_G(G) \cap \mathcal{R}_G(E)} \in \text{Min } \mathcal{R}_G(E)$. The equality follows. \square

Lemma 2.2. *Let $E \subset G \simeq R^e$, $e > 0$, be an R -module. Let $\mathfrak{p} \in \text{Spec}(R)$ and set $\mathcal{P} = \mathfrak{p}\mathcal{R}_G(G) \cap \mathcal{R}_G(E)$. Then*

$$\mathcal{R}_G(E)/\mathcal{P} \simeq \mathcal{R}_{\overline{G}}(\overline{E}),$$

where $\overline{R} = R/\mathfrak{p}$, $\overline{G} = G/\mathfrak{p}G \simeq G \otimes_R \overline{R}$, $\overline{E} = (E + \mathfrak{p}G)/\mathfrak{p}G \subset \overline{G}$.

Proof. Let $\pi: G \rightarrow \overline{G}$ be the canonical epimorphism. Since $\pi(E) = (E + \mathfrak{p}G)/\mathfrak{p}G = \overline{E}$, the map $\pi': E \rightarrow \overline{E}$ defined by $\pi'(z) = \pi(z)$ for every $z \in E$ is an epimorphism. Hence $\mathcal{S}(\pi'): \mathcal{S}_R(E) \rightarrow \mathcal{S}_{\overline{R}}(\overline{E})$ is also an epimorphism. Therefore there exists an R -epimorphism $\rho: \mathcal{R}_G(E) \rightarrow \mathcal{R}_{\overline{G}}(\overline{E})$. Moreover, if $\iota: \mathcal{R}_{\overline{G}}(\overline{E}) \hookrightarrow \mathcal{R}_{\overline{G}}(\overline{G})$ denotes the natural inclusion and $\lambda: \mathcal{R}_G(G) \rightarrow \mathcal{R}_G(G)/\mathfrak{p}\mathcal{R}_G(G) \simeq \mathcal{S}_{\overline{R}}(\overline{G}) = \mathcal{R}_{\overline{G}}(\overline{G})$ denotes the canonical epimorphism, then

$$\lambda|_{\mathcal{R}_G(E)} = \iota \circ \rho.$$

It follows that

$$\mathcal{P} = \mathfrak{p}\mathcal{R}_G(G) \cap \mathcal{R}_G(E) = \ker \lambda|_{\mathcal{R}_G(E)} = \ker(\iota \circ \rho) = \rho^{-1}(\ker \iota) = \ker \rho$$

and so

$$\mathcal{R}_G(E)/\mathcal{P} \simeq \mathcal{R}_{\overline{G}}(\overline{E})$$

as claimed. \square

Our bound for the analytic spread of a module will be a consequence of the following expression of the dimension of the Rees algebra.

Proposition 2.3. *Let $E \subset G \simeq R^e$, $e > 0$, be an R -module. Then*

$$\dim \mathcal{R}_G(E) = \max\{\dim R/\mathfrak{p} + \text{rank}(E + \mathfrak{p}G/\mathfrak{p}G) \mid \mathfrak{p} \in \text{Min } R\} \leq d + e.$$

Proof. We have, by the previous lemmas,

$$\begin{aligned} \dim \mathcal{R}_G(E) &= \max\{\dim \mathcal{R}_G(E)/\mathcal{P} \mid \mathcal{P} \in \text{Min } \mathcal{R}_G(E)\} \\ &= \max\{\dim \mathcal{R}_{G/\mathfrak{p}G}(E + \mathfrak{p}G/\mathfrak{p}G) \mid \mathfrak{p} \in \text{Min } R\}. \end{aligned}$$

On the other hand, for each $\mathfrak{p} \in \text{Min } R$, $\overline{R} = R/\mathfrak{p}$ is a domain and so $\overline{E} = E + \mathfrak{p}G/\mathfrak{p}G \subseteq G/\mathfrak{p}G = \overline{G} \simeq \overline{R}^e$ is a finitely generated torsionfree R -module having rank $r \leq e$. Hence, in this case we have

$$\dim \mathcal{R}_{G/\mathfrak{p}G}(E + \mathfrak{p}G/\mathfrak{p}G) = \dim \overline{R} + \text{rank } \overline{E} \leq d + e$$

and the result follows. \square

Remark 2.4. Observe that the above result and its proof are also valid for any Noetherian ring R , not necessarily local.

As in the ideal case, we may define the fiber cone (or special fiber) of E as $\mathcal{F}_G(E) := \mathcal{R}_G(E)/\mathfrak{m}\mathcal{R}_G(E)$, and the analytic spread of E as $\ell_G(E) := \dim \mathcal{F}_G(E)$. From the above proposition we get the following bound for the analytic spread.

Corollary 2.5. *Let $E \subset G \simeq R^e$, $e > 0$, be an R -module. Assume $d > 0$. Then*

$$\ell_G(E) \leq d + e - 1.$$

Proof. By definition

$$\ell_G(E) = \dim \mathcal{F}_G(E) = \dim \mathcal{R}_G(E)/\mathfrak{m}\mathcal{R}_G(E).$$

If $\mathfrak{m}\mathcal{R}_G(E) \subseteq \mathcal{P}$ for some $\mathcal{P} \in \text{Min } \mathcal{R}_G(E)$, then

$$\mathfrak{m} = \mathfrak{m}\mathcal{R}_G(E) \cap R \subseteq \mathcal{P} \cap R = \mathfrak{p}\mathcal{R}_G(G) \cap \mathcal{R}_G(E) \cap R = \mathfrak{p}\mathcal{R}_G(G) \cap R = \mathfrak{p}$$

where $\mathfrak{p} \in \text{Min } R$ (by Lemma 2.1). But $\dim R > 0$, that is $\mathfrak{m} \notin \text{Min } R$, and so $\mathfrak{m}\mathcal{R}_G(E)$ is not contained in any minimal prime of $\mathcal{R}_G(E)$. It follows that $\text{ht } \mathfrak{m}\mathcal{R}_G(E) > 0$ and so

$$\ell_G(E) = \dim \mathcal{R}_G(E)/\mathfrak{m}\mathcal{R}_G(E) < \dim \mathcal{R}_G(E) \leq d + e,$$

proving the asserted inequality. \square

Although in this paper we are not going to use the theory of reductions of modules, it is worthwhile to point out that, under suitable conditions similar to the ideal case, the analytic spread of an R -module $E \subset G$ coincides with the minimal number of generators of any minimal reduction of E . In particular, and if E has rank, it also holds that $\text{rank } E \leq \ell_G(E)$, see [17, Proposition 2.2].

3. THE ASYMPTOTIC BEHAVIOR OF $\text{depth } G_n/E_n$

Our aim in this section is to prove that $\text{depth } G_n/E_n$ takes a constant value for large n . For that we shall need the following technical lemma.

Lemma 3.1. *Let $E \subsetneq G \simeq R^e$, $e > 0$, be an R -module and let $a \in \mathfrak{m}$. Then, for every $n \geq 1$,*

- (1) $\overline{E}_n \simeq (E_n + aG_n)/aG_n$;
- (2) $(G_n/E_n)/a(G_n/E_n) \simeq \overline{G}_n/\overline{E}_n$;
- (3) $[\mathcal{F}_{\overline{G}}(\overline{E})]_n \simeq E_n/\mathfrak{m}E_n + (aG_n \cap E_n)$;

where $\overline{E} = E + aG/aG \subset \overline{G} = G/aG$.

Proof. Clearly, $\overline{E} \neq \overline{G}$ (by Nakayama's Lemma) since $E \neq G$ and $a \in \mathfrak{m}$. Moreover, for each $n \geq 1$, we have the commutative diagram

$$\begin{array}{ccccc} \mathcal{S}(\overline{E})_n & \longrightarrow & \mathcal{S}(\overline{G})_n & = & \overline{G}_n \\ \simeq & & \simeq & & \\ \mathcal{S}(E)_n \otimes_R R/(a) & \longrightarrow & \mathcal{S}(G)_n \otimes_R R/(a) & = & G_n \otimes_R R/(a) \\ \simeq & & \simeq & & \\ \mathcal{S}(E)_n/a\mathcal{S}(E)_n & \longrightarrow & \mathcal{S}(G)_n/a\mathcal{S}(G)_n & = & G_n/aG_n \end{array}$$

Hence, by the definition of the n -th Rees powers, it follows that

$$\overline{E}_n \simeq (E_n + aG_n)/aG_n,$$

and (1) is proved. Moreover,

$$\begin{aligned} (G_n/E_n)/a(G_n/E_n) &\simeq G_n/(E_n + aG_n) \\ &\simeq (G_n/aG_n)/((E_n + aG_n)/aG_n) \\ &\simeq \overline{G}_n/\overline{E}_n \end{aligned}$$

and (2) is proved. As for (3) we have

$$\begin{aligned} [\mathcal{F}_{\overline{G}}(\overline{E})]_n &= \overline{E}_n/\mathfrak{m}\overline{E}_n \simeq ((E_n + aG_n)/aG_n)/\mathfrak{m}((E_n + aG_n)/aG_n) \\ &\simeq (E_n + aG_n)/(\mathfrak{m}E_n + aG_n) \simeq E_n/((\mathfrak{m}E_n + aG_n) \cap E_n) \\ &\simeq E_n/(\mathfrak{m}E_n + (aG_n \cap E_n)) \end{aligned}$$

- the last isomorphism by the modular law. \square

Remark 3.2. The lemma above still holds if $a \in \mathfrak{m}$ is replaced by an R -ideal $I \subseteq \mathfrak{m}$.

For each $n \geq 1$, we denote the associated prime ideals of G_n/E_n by $A(n)$. By [12, Theorem 2.1], $A(n)$ is stable for large n .

Theorem 3.3. *Let $E \subsetneq G \simeq R^e$, $e > 0$, be an R -module. Then $\text{depth } G_n/E_n$ takes a constant value for large n .*

Proof. We use induction on the inferior limit ⁽¹⁾

$$\alpha = \liminf_{n \rightarrow \infty} \text{depth } G_n/E_n.$$

If $\alpha = 0$ then there exists $m \gg 0$ such that $\text{depth } G_m/E_m = 0$. Hence $\mathfrak{m} \in \text{Ass } G_m/E_m = A(m)$. But $A(n)$ is stable by large n . Hence $\mathfrak{m} \in A(n)$ for all $n \geq m$, and so $\text{depth } G_n/E_n = 0$ for all $n \geq m$.

Now suppose that $\alpha > 0$. Hence there exists $m \geq 1$ such that $\text{depth } G_n/E_n \neq 0$ for all $n \geq m$, that is $\mathfrak{m} \notin A(n)$ for all $n \geq m$. Therefore, there exists $a \in \mathfrak{m}$ such that $a \notin \bigcup_{\mathfrak{p} \in A(n)} \mathfrak{p}$ for sufficiently large n . It follows that

$$a \notin Z_R(G_n/E_n) = \bigcup_{\mathfrak{p} \in A(n)} \mathfrak{p}$$

and we have

$$\text{depth } (G_n/E_n)/a(G_n/E_n) = \text{depth } G_n/E_n - 1 \quad (n \gg 0).$$

On the other hand, by the previous lemma,

$$(G_n/E_n)/a(G_n/E_n) \simeq_{R/(a)} \overline{G_n}/\overline{E_n}$$

and so

$$\beta = \liminf_{n \rightarrow \infty} \text{depth } \overline{G_n}/\overline{E_n} < \liminf_{n \rightarrow \infty} \text{depth } G_n/E_n = \alpha.$$

By induction hypothesis, $\text{depth } \overline{G_n}/\overline{E_n}$ takes a constant value for $n \gg 0$. Hence the result follows by induction. \square

In the following, we shall denote by $\text{depth}(G, E)$ this asymptotic constant value of $\text{depth } G_n/E_n$.

4. THE ASYMPTOTIC BEHAVIOR OF THE ANALYTIC SPREAD OF A MODULE

In this section we prove our result extending to modules the Burch's inequality, see [4]. To do this, we need the following lemmas.

Lemma 4.1. *Let $E \subsetneq G \simeq R^e$, $e > 0$, be an R -module, and assume $d > 0$. Then, for all n ,*

$$\bigoplus_{i \geq 1} \left[\sqrt{\text{ann}_{\mathcal{R}_G(E)}(\mathcal{F}_G(E))} \right]_i = \bigoplus_{i \geq 1} \left[\sqrt{\text{ann}_{\mathcal{R}_G(E)}(\bigoplus_{m \geq n} E_m / \mathfrak{m} E_m)} \right]_i.$$

¹The inferior limite is by definition the smallest of the sublimits.

Proof. The inclusion \subseteq is clear. For the other one it suffices to see that for any homogeneous element $a \in E_h$ ($h > 0$) such that $aE_m \subseteq \mathfrak{m}E_{hm}$ for any $m \geq n$, there exists $s > 0$ with $a^s \in \mathfrak{m}R_G(E)$. Let $s > 0$ such that $(s-1)h \geq n$. Then, $a^s = aa^{s-1} \subseteq aE_{(s-1)h} \subseteq \mathfrak{m}E_{sh}$. Hence $a^s \in \mathfrak{m}R_G(E)$ and the lemma follows. \square

Remark 4.2. Let $E \subsetneq G \simeq R^e$, $e > 0$, be an R -module. Suppose that $d > 0$ and $\text{depth}(G, E) > 0$. Then there exists $a \in \mathfrak{m}$ such that $a \notin Z_R(G_n/E_n)$ for all $n \gg 0$ and $a \notin \mathfrak{p}$ for all $\mathfrak{p} \in \text{Min } R$.

Proof. Since $\text{depth}(G, E) > 0$, then $\alpha = \liminf_{n \rightarrow \infty} \text{depth } G_n/E_n > 0$ and, as in the proof of Theorem 3.3, $\mathfrak{m} \notin A(n)$ for all $n \gg 0$. Moreover, $\mathfrak{m} \notin \text{Min } R$, since $\dim R > 0$. Hence

$$\mathfrak{m} \notin \left(\bigcup_{n \gg 0} A(n) \right) \cup \text{Min } R,$$

It follows that there exists $a \in \mathfrak{m}$ such that $a \notin Z_R(G_n/E_n)$ for all $n \gg 0$ and $a \notin \mathfrak{p}$ for all $\mathfrak{p} \in \text{Min } R$, as asserted. \square

Lemma 4.3. Let $E \subsetneq G \simeq R^e$, $e > 0$, be an R -module. Suppose that $d > 0$ and $\text{depth}(G, E) > 0$. Then

$$\ell_G(E) = \ell_{\overline{G}}(\overline{E}),$$

where $\overline{E} = E + aG/aG$, $\overline{G} = G/aG$ and $a \in \mathfrak{m} \setminus \bigcup_{n \gg 0} Z_R(G_n/E_n)$.

Proof. Since $\text{depth}(G, E) > 0$, we may choose an $a \in \mathfrak{m} \setminus \bigcup_{n \gg 0} Z_R(G_n/E_n)$. By Lemma 3.1, we have, for each n ,

$$[\mathcal{F}_{\overline{G}}(\overline{E})]_n \simeq E_n/(\mathfrak{m}E_n + (aG_n \cap E_n)).$$

Now, since $a \in \mathfrak{m}$ and a is regular with respect to G_n/E_n for all sufficiently large n

$$(E_n :_{G_n} a) = \{z \in G_n \mid az \in E_n\} = E_n$$

for all $n \gg 0$. Hence, for $n \gg 0$, $aG_n \cap E_n = aE_n \subseteq \mathfrak{m}E_n$, and so $\mathfrak{m}E_n + (aG_n \cap E_n) = \mathfrak{m}E_n$. It follows that

$$[\mathcal{F}_{\overline{G}}(\overline{E})]_n \simeq [\mathcal{F}_G(E)]_n$$

for $n \gg 0$. Therefore, by the Lemma 4.1, we have for $n \gg 0$

$$\begin{aligned} \bigoplus_{i \geq 1} \left[\sqrt{\text{ann}_{\mathcal{R}_G(E)} \mathcal{F}_G(E)} \right]_i &= \bigoplus_{i \geq 1} \left[\sqrt{\text{ann}_{\mathcal{R}_G(E)} (\oplus_{m \geq n} \mathcal{F}_G(E)_m)} \right]_i \\ &= \bigoplus_{i \geq 1} \left[\sqrt{\text{ann}_{\mathcal{R}_G(E)} (\oplus_{m \geq n} [\mathcal{F}_{\overline{G}}(\overline{E})]_m)} \right]_i \\ &= \bigoplus_{i \geq 1} \left[\sqrt{\text{ann}_{\mathcal{R}_{\overline{G}}(\overline{E})} (\oplus_{m \geq n} [\mathcal{F}_{\overline{G}}(\overline{E})]_m)} \right]_i \\ &= \bigoplus_{i \geq 1} \left[\sqrt{\text{ann}_{\mathcal{R}_{\overline{G}}(\overline{E})} \mathcal{F}_{\overline{G}}(\overline{E})} \right]_i. \end{aligned}$$

Moreover, since $a \in \mathfrak{m}$,

$$\begin{aligned} [\text{ann}_{\mathcal{R}_G(E)} \mathcal{F}_G(E)]_0 &= \text{ann}_R R/\mathfrak{m} = \text{ann}_{R/(a)}(R/(a))/\mathfrak{m}(R/(a)) \\ &= [\text{ann}_{\mathcal{R}_{\overline{G}}(\overline{E})} \mathcal{F}_{\overline{G}}(\overline{E})]_0. \end{aligned}$$

Now, since

$$\begin{aligned} \ell_G(E) &= \dim \mathcal{F}_G(E) = \dim(\mathcal{R}_G(E)/\text{ann}_{\mathcal{R}_G(E)} \mathcal{F}_G(E)) \\ &= \dim \left(\mathcal{R}_G(E)/\sqrt{\text{ann}_{\mathcal{R}_G(E)} \mathcal{F}_G(E)} \right), \end{aligned}$$

the result follows. \square

As in the case of ideals, by using the asymptotic value of $\text{depth } G_n/E_n$ one can obtain a slightly better bound than the original one in Burch's inequality. Namely,

Theorem 4.4. *Let $E \subsetneq G \simeq R^e$, $e > 0$, be an R -module. Assume $d > 0$. Then*

$$\ell_G(E) \leq d + e - 1 - \text{depth}(G, E) \leq d + e - 1 - \inf_{n \geq 1} \text{depth } G_n/E_n.$$

Proof. We use induction on $\beta = \text{depth}(G, E)$ to prove the first inequality. If $\beta = 0$, we apply Corollary 2.5. Now, suppose that $\beta > 0$. Let $a \in \mathfrak{m}$ such that $a \notin Z_R(G_n/E_n)$ for all $n \gg 0$ and $a \notin \mathfrak{p}$ for all $\mathfrak{p} \in \text{Min } R$, which exists by Remark 4.2. By the lemma above

$$\ell_G(E) = \ell_{\overline{G}}(\overline{E}),$$

where $\overline{E}, \overline{G}$ are as in Lemma 3.1. Moreover, since $a \notin Z_R(G_n/E_n)$ for all $n \gg 0$,

$$\text{depth } G_n/E_n = \text{depth}(G_n/E_n)/a(G_n/E_n) + 1 = \text{depth } \overline{G}_n/\overline{E}_n + 1,$$

for $n \gg 0$. Hence

$$\text{depth}(\overline{G}, \overline{E}) = \text{depth}(G, E) - 1.$$

Moreover, $\overline{G} \simeq (R/(a))^e$. Further, since $a \notin \mathfrak{p}$ for all $\mathfrak{p} \in \text{Min } R$

$$\dim R/(a) = \dim R - 1 = d - 1.$$

Now by induction,

$$\ell_{\overline{G}}(\overline{E}) \leq (d - 1) + e - 1 - \text{depth}(\overline{G}, \overline{E}).$$

It follows that

$$\ell_G(E) = \ell_{\overline{G}}(\overline{E}) \leq d + e - 1 - \text{depth}(G, E)$$

as asserted. Finally, since

$$\text{depth}(G, E) \geq \inf_{n \geq 1} \text{depth } G_n/E_n$$

the result follows. \square

In the case of ideals, it is easy to prove that if $a \notin \bigcup_{n \geq 1} Z_R(R/I^n)$ then

$$\mathcal{R}(I/aI) \simeq \mathcal{R}((I + aR)/aR) \simeq \mathcal{R}(I) \otimes_R R/aR \simeq \mathcal{R}(I)/a\mathcal{R}(I).$$

Furthermore, if $a \in \mathfrak{m} \setminus \bigcup_{n \geq 1} Z_R(R/I^n)$ then

$$\mathcal{F}(I/aI) \simeq \mathcal{R}(I) \otimes_R R/aR \otimes_R R/\mathfrak{m} \simeq \mathcal{R}(I) \otimes_R R/\mathfrak{m} = \mathcal{F}(I).$$

For modules we can deduce the following.

Proposition 4.5. *Let $E \subsetneq G \simeq R^e$, $e > 0$, be an R -module. Assume $d > 0$. If $a \in \mathfrak{m} \setminus \bigcup_{n \geq 1} Z_R(G_n/E_n)$ then*

- (1) $aG_n \cap E_n = aE_n \subseteq \mathfrak{m}E_n$ for all $n \geq 1$;
- (2) $\mathcal{R}_{\overline{G}}(\overline{E}) \simeq \mathcal{R}_G(E) \otimes_R R/aR \simeq \mathcal{R}_G(E)/a\mathcal{R}_G(E)$;
- (3) $\mathcal{F}_{\overline{G}}(\overline{E}) \simeq \mathcal{F}_G(E)$.

Proof. Let $a \in \mathfrak{m} \setminus \bigcup_{n \geq 1} Z_R(G_n/E_n)$. Hence, as in the proof of Lemma 4.3, we have $aG_n \cap E_n = aE_n \subseteq \mathfrak{m}E_n$ for all $n \geq 1$. In particular, $aG \cap E = aE$. It follows that

$$E/aE = E/(aG \cap E) \simeq (E + aG)/aG \simeq \overline{E}.$$

For the assertions (2) and (3), we use Lemma 3.1. In fact, for all $n \geq 1$

$$\overline{E}_n = (E_n + aG_n)/aG_n \simeq E_n/(aG_n \cap E_n) \simeq E_n/aE_n = [\mathcal{R}_G(E)/a\mathcal{R}_G(E)]_n$$

and, since $a \in \mathfrak{m}$,

$$\begin{aligned} [\mathcal{F}_{\overline{G}}(\overline{E})]_n &= [\mathcal{F}_{\overline{G}}(\overline{E})]_n \simeq E_n/(\mathfrak{m}E_n + (aG_n \cap E_n)) \\ &\simeq E_n/(\mathfrak{m}E_n + aE_n) = E_n/\mathfrak{m}E_n = [\mathcal{F}_G(E)]_n. \end{aligned}$$

The result follows. \square

5. THE CASE $\mathcal{R}_G(E)$ BEING COHEN-MACAULAY

Next we shall prove that, in the case where the Rees algebra of a finitely generated torsionfree R -module having rank is Cohen-Macaulay, the Burch's inequality is in fact an equality.

Given a finitely generated module E over a Noetherian ring R and an R -ideal I such that $IE \neq E$

$$(2) \quad \text{depth}_I E = \inf\{i \in \mathbb{N}_0 \mid H_I^i(E) \neq 0\},$$

where $H_I^i(E)$ denotes the i -th local cohomology module of E with respect to I (cf. [3, Theorem 6.2.7]). Moreover, if $\varphi: R \rightarrow S$ is a homomorphism of Noetherian (graded) rings and I is a (homogeneous) ideal of R and E is a finitely generated S -module

$$(3) \quad H_I^i(E) \simeq H_{IS}^i(E),$$

(cf. [10, Corollary 35.20]).

Lemma 5.1. *Let $E \subsetneq G \simeq R^e$, $e > 0$, be an R -module. Then*

$$\text{grade } \mathfrak{m}\mathcal{R}_G(E) = \inf_{n \geq 0} \text{depth } E_n.$$

Proof. We have, by (2)

$$\text{grade } \mathfrak{m}\mathcal{R}_G(E) = \text{depth}_{\mathfrak{m}\mathcal{R}_G(E)}(\mathcal{R}_G(E)) = \inf\{i \in \mathbb{N}_0 \mid H_{\mathfrak{m}\mathcal{R}_G(E)}^i(\mathcal{R}_G(E)) \neq 0\}$$

since $\mathcal{R}_G(E)$ is a Noetherian ring. Moreover, since $\mathcal{R}_G(E)$ is an R -module, by (3)

$$H_{\mathfrak{m}\mathcal{R}_G(E)}^i(\mathcal{R}_G(E)) = H_{\mathfrak{m}}^i(\mathcal{R}_G(E)) = H_{\mathfrak{m}}^i(\bigoplus_{n \geq 0} E_n) = \bigoplus_{n \geq 0} H_{\mathfrak{m}}^i(E_n)$$

by [3, Theorem 3.4.10]. Therefore

$$H_{\mathfrak{m}\mathcal{R}_G(E)}^i(\mathcal{R}_G(E)) \neq 0 \iff \exists m \geq 0 : H_{\mathfrak{m}}^i(E_m) \neq 0.$$

Now, suppose that $\text{grade } \mathfrak{m}\mathcal{R}_G(E) = j$. Then there exists an $m \geq 0$ such that $H_{\mathfrak{m}}^j(E_m) \neq 0$, and so

$$\inf_{n \geq 0} \text{depth } E_n \leq \text{depth } E_m \leq j = \text{grade } \mathfrak{m}\mathcal{R}_G(E).$$

On the other hand, suppose that $\text{depth } E_s = \inf_{n \geq 0} \text{depth } E_n$ and suppose that $\text{depth } E_s = j$. Then $H_{\mathfrak{m}}^j(E_s) \neq 0$, and so $H_{\mathfrak{m}\mathcal{R}_G(E)}^j(\mathcal{R}_G(E)) \neq 0$. It follows that

$$\text{grade } \mathfrak{m}\mathcal{R}_G(E) \leq j = \text{depth } E_s = \inf_{n \geq 0} \text{depth } E_n,$$

and the equality follows. \square

We have the following bound for the depth of the Rees algebra of a module.

Proposition 5.2. *Let $E \subsetneq G \simeq R^e$, $e > 0$, be an R -module. Then*

$$\text{depth } \mathcal{R}_G(E) \leq \inf_{n \geq 0} \text{depth } E_n + \ell_G(E).$$

Proof. By the lemma above we have

$$\begin{aligned} \inf_{n \geq 0} \text{depth } E_n &= \text{grade } \mathfrak{m}\mathcal{R}_G(E) \\ &\geq \text{depth } \mathcal{R}_G(E) - \dim \mathcal{R}_G(E)/\mathfrak{m}\mathcal{R}_G(E) \\ &= \text{depth } \mathcal{R}_G(E) - \ell_G(E) \end{aligned}$$

by [14, Th. 17.1], proving the inequality. \square

The next result was originally proved by D. Eisenbud and C. Huneke [8] in the ideal case.

Corollary 5.3. *Assume that R is Cohen-Macaulay and $d > 0$. Let $E \subset G \simeq R^e$, $e > 0$, be an R -module with rank but not free. If $\mathcal{R}_G(E)$ is Cohen-Macaulay then*

$$\ell_G(E) = d + e - 1 - \inf_{n \geq 1} \text{depth } G_n/E_n.$$

Proof. By Theorem 4.4

$$\ell_G(E) \leq d + e - 1 - \inf_{n \geq 1} \text{depth } G_n/E_n.$$

By the proposition above, and since R and $\mathcal{R}_G(E)$ are both Cohen-Macaulay, we have

$$\begin{aligned} \ell_G(E) &\geq \text{depth } \mathcal{R}_G(E) - \inf_{n \geq 0} \text{depth } E_n \\ &= \dim \mathcal{R}_G(E) - (\inf_{n \geq 0} \text{depth } G_n/E_n + 1) \\ &= d + e - 1 - \inf_{n \geq 0} \text{depth } G_n/E_n. \end{aligned}$$

The equality follows. \square

Our final result in this section may be useful for induction arguments.

Corollary 5.4. *Assume that R is Cohen-Macaulay and $d > 0$. Let $E \subset G \simeq R^e$, $e > 0$, be an R -module with rank but not free. If $\mathcal{R}_G(E)$ is Cohen-Macaulay and $\ell_G(E) < d + e - 1$ then there exists $a \in \mathfrak{m}$ such that*

- (1) $aG_n \cap E_n = aE_n \subseteq \mathfrak{m}E_n$ for all $n \geq 1$;
- (2) $\mathcal{R}_{\overline{G}}(\overline{E}) \simeq \mathcal{R}_G(E) \otimes_R R/aR \simeq \mathcal{R}_G(E)/a\mathcal{R}_G(E)$;
- (3) $\mathcal{F}_{\overline{G}}(\overline{E}) \simeq \mathcal{F}_G(E)$.

Proof. Since $\ell_G(E) < d + e - 1$ we have by Corollary 5.3 that $\inf_{n \geq 1} \text{depth } G_n/E_n > 0$. Therefore, $\mathfrak{m} \notin A(n) = \text{Ass}(G_n/E_n)$ for every $n \geq 1$, and so there exists $a \in \mathfrak{m} \setminus \bigcup_{n \geq 1} Z_R(G_n/E_n)$. Now apply Corollary 4.5. \square

6. A CRITERIA FOR COMPLETE INTERSECTION

Let $E \subsetneq G \simeq R^e$, $e > 0$, be an R -submodule of G , with rank e but not free. Following A. Simis, B. Ulrich, and W. V. Vasconcelos [17] we say that E is an ideal module if the double dual E^{**} is free. Ideal modules provide a natural extension of several notions in analogy to the case of ideals. Namely, if E is an ideal module we define the deviation of E by $d(E) = \mu(E) - e + 1 - \text{ht } F_e(E)$ and the analytic deviation of E by $\text{ad}(E) = \ell_G(E) - e + 1 - \text{ht } F_e(E)$, where $\mu(\cdot)$ denotes the minimal number of generators and $F_e(E)$ is the e -th Fitting invariant of E . Similarly to the ideal case, one has that the inequalities $d(E) \geq \text{ad}(E) \geq 0$ hold for any ideal module E , see [6, Proposition 4.2.1]. We then say that an ideal module E is a complete intersection if $d(E) = 0$ and equimultiple if $\text{ad}(E) = 0$. Obviously, complete intersection ideal modules are equimultiple. We also say that an ideal module E is generically a complete intersection if $\mu(E_{\mathfrak{p}}) = \text{ht } F_e(E) + e - 1$ for all minimal prime ideals $\mathfrak{p} \in \text{Min } R/F_e(E)$.

Remark 6.1. Our definitions of deviation and analytic deviation slightly differ from those in [17] since there it is used $\text{grade } F_e(E)$ instead of $\text{ht } F_e(E)$. Of course, they coincide if R is Cohen-Macaulay.

Our aim in this section is to extend to ideal modules the famous criteria by R. C. Cowsik and M. V. Nori [7] for an ideal to be a complete intersection. First, and as a consequence of the Burch's inequality, we have the following criteria for an ideal module E to be equimultiple in terms of the behaviour of depths of its Rees powers.

Corollary 6.2. *Let $E \subsetneq G \simeq R^e$, $e > 0$, be an ideal module. If $\text{depth } G_n/E_n = d - \text{ht } F_e(E)$ for infinitely many n , then E is equimultiple.*

Proof. By assumption, $\text{depth}(G, E) = d - \text{ht } F_e(E)$. Applying Burch's inequality, we obtain

$$\ell_G(E) \leq d + e - 1 - \text{depth}(G, E) = \text{ht } F_e(E) + e - 1 \leq \ell_G(E)$$

proving that E is equimultiple. \square

The following lemma extends to ideal modules a result which has been very fruitful in the ideal case. We refer to [6, Proposition 4.2.14] for its proof. It follows the same lines as in the case of ideals by using the theory of reductions of modules.

Lemma 6.3. *Assume that R is Cohen-Macaulay and let E be an ideal module. Suppose that E is generically a complete intersection. Then E is a complete intersection if and only if E is equimultiple.*

Now we are ready to prove for ideal modules the extension of the criteria by R. C. Cowsik and M. V. Nori. We obtain it as consequence of our version for modules of the Burch's inequality. In this way, we even get a slightly better version of the criteria, in the same way as M. Brodmann [2] did for ideals.

Theorem 6.4. *Let R be a cohen-Macaulay local ring, $\dim R = d > 0$, and let $E \subseteq G \simeq R^e$ be an ideal module having rank $e > 0$. If E is generically a complete intersection then the following are all equivalent:*

- (1) E is a complete intersection;
- (2) G_n/E_n are Cohen-Macaulay for all $n > 0$;
- (3) G_n/E_n are Cohen-Macaulay for infinitely many n .

Proof. (1) \Rightarrow (2) This was already proved by D. Katz and C. Naude [12, Proposition 3.3] (in fact, they proved a stronger result: That G_n/E_n are perfect of dimension $d - \text{ht } F_e(E)$ for all $n \geq 1$).

(2) \Rightarrow (3) is immediate.

(2) \Rightarrow (3) In virtue of Lemma 6.3 it is enough to show that E is equimultiple. Since E is generically a complete intersection, we have by the result of D. Katz and C. Naude that $\dim G_n/E_n = d - \text{ht } F_e(E)$ for all $n \geq 1$. Now, by assumption

$$\text{depth } G_n/E_n = \dim G_n/E_n = d - \text{ht } F_e(E)$$

for infinitely many n , hence by Corollary 6.2 E is equimultiple. \square

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CENTRO DE ANÁLISE MATEMÁTICA, GEOMETRIA E SISTEMAS DINÂMICOS, INSTITUTO SUPERIOR TÉCNICO, AV. ROVISCO PAIS, 1049-001 LISBOA, PORTUGAL

E-mail address: matalrbc@univ-ab.pt

UNIVERSIDADE ABERTA, RUA FERNÃO LOPES 2º DTO, 1000-132 LISBOA, PORTUGAL

E-mail address: matalrbc@univ-ab.pt

DEPARTAMENT D'ÀLGEBRA I GEOMETRIA, UNIVERSITAT DE BARCELONA, GRAN VIA 585, E-08007 BARCELONA, SPAIN

E-mail address: zarzuela@mat.ub.es